# Analytic Calibration of Black-Karasinski Short Rate Model for Low Rates

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**Abstract**

We consider calibration of the Black-Karasinski short rate model to a given interest rate (or credit intensity) term structure, conducting an asymptotic analysis in the limit of low rates. We base our analysis on the perturbation expansion approach proposed by Turfus and Schubert (2016). We calibrate the model so as to give consistent representation of the probability distribution of $T$-maturity zero coupon bond prices at all times up to maturity.

## 1 Introduction

As pointed out recently by Turfus (2016b), the Black-Karasinski lognormal volatility short rate model is known to have many attractive features as an interest rate model; however its most significant perceived drawback since it was first proposed 25 years ago has been its lack of tractible analytic formulae for conditional zero coupon bond prices or caplets. This was recently addressed by Turfus (2016b) using a perturbation expansion based on an assumption of small volatility. It was proposed from the form of the expansion derived that, even when the lognormal volatility is not particularly "small", the caplet formulae ought still to be reliable, provided that the rates themselves are not too large.

It is often however wished to use the Black-Karasinski interest rate model as a credit intensity model, given its ability to ensure positive rates. This feature makes it more attractive than the otherwise popular (because relatively tractible) normal volatility short rate model of Hull and White (1990). But it is the square root model of Cox, Ingersoll and Ross (1985) which tends to be most favoured for credit intensity modelling because it is both tractible and can be configured to ensure positive rates. None the less, as has been pointed out by Brigo and Mercurio (2006), it too has a significant downside as a credit model insofar as it is limited in its ability to calibrate to markets with implied lognormal volatility in excess of about 40–50%. For this reason we propose below a perturbation expansion approach to allow calibration of a Black-Karasinski model to a given forward curve (interest rate or credit intensity) subject to the assumption that the level of the forward curve remains low, but without any assumption about the level of the volatility.

Having set out our model and modelling assumptions in §2, we proceed by calculating in §3 the theoretical distribution under our model of $T$-maturity zero coupon bonds at arbitrary future times $t$. By applying a condition of the consistency of the implied present values with the market-specified forward curve, we are able to tie down the model to a unique calibration in an asymptotic sense of errors in bond prices being formally $O(\epsilon_r^3)$, where $\epsilon_r$ is a parameter capturing the "smallness" of the short rates under consideration.[1] It is proposed that, even for fairly stressed credit intensity rates, this ought to provide sufficient accuracy for many practical purposes, in particular risk management, which is where a great deal of the focus is in credit modelling in the finance industry at present.

[1] Or, equivalently, the relative error in the calculated rates $= O(\epsilon_r^2)$

## 2 Modelling Assumptions

We consider short rate processes representing the interest short rate to be driven by a Black-Karasinski model. We shall find it convenient to work with a reduced variable $y_t$ satisfying the following canonical Ornstein-Uhlenbeck process:

$$dy_t = -\alpha_r y_t dt + \sigma_r(t)\, dW_t \tag{1}$$

where $dW_t$ is a standard Brownian motion under the martingale measure. This is taken to be related to the interest short rate $r_t$ by

$$r_t = r^*(t) e^{y_t - \frac{1}{2} I_r(t_0,t)} \tag{2}$$

where

$$I_r(t_1,t_2) := \int_{t_1}^{t_2} e^{-2\alpha_r(t_2-s)} \sigma_r^2(s)\, ds. \tag{3}$$

We take $t_0$ to be the initial observation date with, by assumption, $y_{t_0} = 0$. The function $r^*(t)$ is determined by calibration but is constrained for consistency to tend in the zero volatility limit to the forward interest rate $\overline{r}(t)$. The formal no-arbitrage constraint which determines this function is as follows

$$E\left[e^{-\int_{t_0}^{t} r_s\, ds}\right] = e^{-\int_{t_0}^{t} \overline{r}(s)\, ds} \tag{4}$$

under the martingale measure (money market numéraire) for $t_0 < t \le T_m$, where $T_m$ is the longest maturity date for which the model is calibrated. We proceed by writing the (stochastic) price at time $t$ of a unit cash flow at time $T$ as $f_T(y_t, t)$. We then look to price these cash flows and so to determine the conditions necessary to satisfy Eq. 4.

For future notational convenience, we also introduce at this stage the deterministic discount factors

$$D(t_1,t_2) = e^{-\int_{t_1}^{t_2} \overline{r}(s)\, ds} \tag{5}$$

## 3 Calibration of Model to Zero Coupon Bonds

We deduce by standard means that the $T$-maturity zero coupon bond price $f_T(y,t)$ will be governed under the money market numéraire by the following backward diffusion equation:

$$\frac{\partial f_T}{\partial t} - \alpha_r y \frac{\partial f_T}{\partial y} + \tfrac{1}{2}\sigma_r^2(t) \frac{\partial^2 f_T}{\partial y^2} = r^*(t) e^y f_T, \tag{6}$$

the r.h.s. being simply $f_T$ multiplied by our assumed functional representation of $r_t$.

In the absence of exact closed form solutions to Eq. 6, we seek an approximate solution under a low interest rate assumption, i.e. we rescale $\overline{r}(t)$ by a non-dimensionalised asymptotic parameter $\epsilon_r$ defined by

$$\epsilon_r := \frac{\int_{t_0}^{T_m} \overline{r}(t)\, dt}{\alpha_r(T_m - t_0)}$$

and a scaled forward rate

$$\tilde{r}(t) := \epsilon_r^{-1} \overline{r}(t)$$

where $\tilde{r}(t)$ is $O(1)$ as $\epsilon_r \to 0$. On this basis we expect the following asymptotic form for $r^*(t)$:

$$r^*(t) = \epsilon_r r_1^*(t) + \epsilon_r^2 r_2^*(t) + O(\epsilon_r^3) \tag{7}$$

We can then rewrite Eq. 6 as

$$\mathcal{L}[f_T(y,t)] = \left(\epsilon_r \left(r_1^*(t) e^{y - \frac{1}{2} I_r(t_0,t)} - \tilde{r}(t)\right) + \epsilon_r^2 r_2^*(t) e^{y - \frac{1}{2} I_r(t_0,t)}\right) f_T(y,t) + O(\epsilon_r^3) \tag{8}$$

where $\mathcal{L}[.]$ is a standard forced diffusion operator given by[2]

$$\mathcal{L}[.] = \frac{\partial}{\partial t} - \alpha_r y \frac{\partial}{\partial y} + \tfrac{1}{2}\sigma_r^2(t) \frac{\partial^2}{\partial y^2} - \overline{r}(t), \tag{9}$$

and the final condition that must be satisfied is

$$f_T(y, T) = 1. \tag{10}$$

To solve for $f_T(y,t)$ we pose an asymptotic expansion

$$f_T(y,t) = f_0(t,T) + \epsilon_r f_1(y,t,T) + \epsilon_r^2 f_2(y,t,T) + O(\epsilon_r^3). \tag{11}$$

At zeroth order we have that $\mathcal{L}[f_0(t)] = 0$ with the trivial solution that

$$f_0(t,T) = D(t,T),$$

which clearly satisfies the required final condition. At first order we have

$$\mathcal{L}[f_1(y,t,T)] = \left( r_1^*(t) e^{y - \frac{1}{2} I_r(t_0,t)} - \tilde{r}(t) \right) f_0(t,T) \tag{12}$$

The final condition Eq. 10 requires that $f_1(y_T, T, T) = 0$. The no-arbitrage condition Eq. 4 introduces the further constraint that $f_1(0, t_0, T) = 0$.

To proceed we observe that Eq. 9 has a Green's function solution given by

$$G(y,t;\eta,v) = D(t,v) H(v-t) \frac{\partial}{\partial \eta} N\left( \frac{\eta - y e^{-\alpha_r(v-t)}}{\sqrt{I_r(t,v)}} \right) \tag{13}$$

where $H(.)$ is the Heaviside step function and $N(.)$ is a standard unit normal distribution. We deduce by standard means:

$$f_1(y,t,T) = -D(t,T) \int_t^T \left( r_1^*(v) \exp\left( e^{-\alpha_r(v-t)} y - \tfrac{1}{2} e^{-2\alpha_r(v-t)} I_r(t_0,t) \right) - \tilde{r}(v) \right) dv.$$

Applying the no-arbitrage condition $f_1(0, t_0, T) = 0$ we conclude we must choose

$$r_1^*(t) = \tilde{r}(t) \tag{14}$$

whence our first order result can be written in terms of the Doléans-Dade exponential function $\mathcal{E}(.)$ as

$$f_1(y_t, t, T) = -D(t,T) \int_t^T \left( \mathcal{E}\left( e^{-\alpha_r(v-t)} y_t \right) - 1 \right) \tilde{r}(v)\, dv. \tag{15}$$

Proceeding in a similar vein, we find at second order

$$\mathcal{L}[f_2(y,t,T)] = \tilde{r}(t) \left( e^{y - \frac{1}{2} I_r(t_0,t)} - 1 \right) f_1(y,t,T) + r_2^*(t) e^{y - \frac{1}{2} I_r(t_0,t)} f_0(t,T). \tag{16}$$

with solution given by

$$\begin{aligned}
&f_2(y_t,t,T) \\
&\quad = -\int_t^T \int_{-\infty}^{\infty} G(y_t,t;\eta,v) \left[ \tilde{r}(v) \left( e^{\eta - \frac{1}{2} I_r(t_0,v)} - 1 \right) f_1(\eta,v,T) + r_2^*(v) e^{\eta - \frac{1}{2} I_r(t_0,v)} f_0(v,T) \right] d\eta\, dv \\
&\quad = D(t,T) \int_t^T \tilde{r}(u) \int_t^u \left( \mathcal{E}\left( e^{-\alpha_r(u-t)} y_t \right) \mathcal{E}\left( e^{-\alpha_r(v-t)} y_t \right) \exp\left( e^{-\alpha_r(u-v)} I_r(t,v) \right) \right. \\
&\qquad\qquad \left. - \mathcal{E}\left( e^{-\alpha_r(v-t)} y_t \right) - \mathcal{E}\left( e^{-\alpha_r(u-t)} y_t \right) + 1 \right) \tilde{r}(v)\, dv\, du \\
&\qquad - D(t,T) \int_t^T r_2^*(u) \mathcal{E}\left( e^{-\alpha_r(u-t)} y_t \right) du
\end{aligned}$$

[2] Note that we have kept $\overline{r}(t)$ in its original unscaled form within the operator definition since this drives only the discount factor scaling, in which context we will not seek to expand in terms of the small parameter.

This clearly satisfies the final condition $f_2(y_T,T,T)=0$. At $t=t_0$ we have

$$f_2(0,t_0,T) = D(t_0,T)\int_{t_0}^{T}\left(\tilde r(u)\int_{t_0}^{u} F^*(v,u)\tilde r(v)\,dv - r_2^*(u)\right)\,du$$

where we have defined

$$F^*(v,u) := \exp\left(e^{-\alpha_r(u-v)}I_r(t_0,v)\right)-1 \tag{17}$$

The initial condition $f_2(0,t_0,T)=0$ then requires us to choose

$$r_2^*(u) = \tilde r(u)\int_{t_0}^{u} F^*(v,u)\tilde r(v)\,dv. \tag{18}$$

Finally we can write[3]

$$\begin{aligned}
f_2(y_t,t,T) = {} & \tfrac{1}{2}D(t,T)\left(\int_t^T\left(\mathcal{E}\left(e^{-\alpha_r(v-t)}y_t\right)-1\right)\tilde r(v)\,dv\right)^2 \\
& + D(t,T)\int_t^T \mathcal{E}\left(e^{-\alpha_r(u-t)}y_t\right)\tilde r(u)\int_t^u F^*(v,u)\left(\mathcal{E}\left(e^{-\alpha_r(v-t)}y_t\right)-1\right)\tilde r(v)\,dv\,du \\
& - D(t,T)\int_t^T \mathcal{E}\left(e^{-\alpha_r(u-t)}y_t\right)\tilde r(u)\int_{t_0}^t F^*(v,u)\,\tilde r(v)\,dv\,du
\end{aligned} \tag{19}$$

Combining all contributions and reverting to unscaled notation, we obtain our main result:

$$\begin{aligned}
f_T(y_t,t) = D(t,T)\Bigg[ & 1-\int_t^T\left(\mathcal{E}\left(e^{-\alpha_r(v-t)}y_t\right)-1\right)\overline{r}(v)\,dv + \tfrac{1}{2}\left(\int_t^T\left(\mathcal{E}\left(e^{-\alpha_r(v-t)}y_t\right)-1\right)\overline{r}(v)\,dv\right)^2 \\
& + \int_t^T \mathcal{E}\left(e^{-\alpha_r(u-t)}y_t\right)\overline{r}(u)\int_t^u F^*(v,u)\left(\mathcal{E}\left(e^{-\alpha_r(v-t)}y_t\right)-1\right)\overline{r}(v)\,dv\,du \\
& - \int_t^T \mathcal{E}\left(e^{-\alpha_r(u-t)}y_t\right)\overline{r}(u)\int_{t_0}^t F^*(v,u)\overline{r}(v)\,dv\,du\Bigg] + O(\epsilon_r^3)
\end{aligned} \tag{20}$$

The other main results are Eqs. 14 and 18 which in unscaled notation can be written

$$r_1^*(u) = \overline{r}(u) \tag{21}$$

$$r_2^*(u) = \overline{r}(u)\int_{t_0}^{u} F^*(v,u)\overline{r}(v)\,dv. \tag{22}$$

This gives the model calibration, accurate to $O(\epsilon_r^2)$.

## References


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[3] We have here used the identity that, by symmetry, $\int_t^T f(v)\int_t^v f(u)\,du\,dv = \frac{1}{2}\left(\int_t^T f(u)\,du\right)^2$.

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